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Large-Amplitude Free Vibrations of Uniform Timoshenko Beams: A Novel Formulation

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Nomenclature

b	=	undetermined coefficient
b_m	=	maximum amplitude
E	=	Young's modulus
I	=	area moment of inertia
k	=	shear correction factor
L	=	length of the beam
m	=	mass of the beam per unit length
P_{cr}	=	buckling load
P_i	=	initial end concentrated load
P_T	=	axial tensile load
r	=	radius of gyration
w	=	lateral displacement
x	=	axial coordinate
β	=	slenderness ratio (=L/r)
λ_b	=	buckling load parameter
λ_f	=	linear frequency parameter with initial load
λ_{f_0}	=	linear frequency parameter without initial load
λ_i	=	initial load parameter
λ_L	=	linear frequency parameter
λ_{NL}	=	nonlinear frequency parameter
λ_T	=	axial tensile load parameter
ν	=	Poisson ratio
ω_f	=	radian frequency with initial load
ω_L	=	linear radian frequency
ω_{NL}	=	nonlinear radian frequency
ω_0	=	radian frequency without initial load

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Introduction

SLENDER (Euler–Bernoulli) and short (Timoshenko) beams are widely used structural members in aerospace and other structures. Although the slender beams are of common occurrence, the short beams are used in landing gear mechanisms, stub adaptors, vented interstages, intertank structures, etc., for achieving functional requirements of the aerospace structural subsystems. These beams are of varying orders of slenderness ratios. If the slenderness ratios are less than 100 (Timoshenko beams), the effects of shear deformation and rotary inertia are to be considered in the analysis of such structural members. The aerospace structures are subjected to severe dynamic loads during the operational phase, leading to large deflections and as such the effect of the geometric nonlinearity is to be considered in the free vibration analysis. A novel method is proposed in this note to obtain the frequency–amplitude relationship which is the basic input for obtaining the nonlinear dynamic response.

The classic work of Woinowsky-Krieger [1] on the large-amplitude free vibrations of hinged–hinged beams inspired many a researcher to develop new continuum and numerical formulations with some assumptions on the formulations and the nature of vibrations to arrive at viable simple formulations. A systematic discussion on the effect of these assumptions and formulations are discussed by Singh et al. [2] and Marur [3]. An exhaustive presentation on the recent developments in the area of nonlinear analysis of structural members, including the large-amplitude free vibrations, is available in the book by Sathyamoorthy [4]. The continuum or numerical formulations discussed in these references are rather complex in nature and are not easily amenable for obtaining simple, accurate, and quick closed-form solutions that would be very attractive to the structural engineering community.

It is the endeavor of the lead author to develop some very simple, and at the same time accurate and reliable, formulations to some complex problems of practical interest in the field of structural mechanics, and successfully develop such a formulation for predicting the fundamental frequency parameter of beams with initial and axial loads [5]. Following this earlier work, the authors have attempted to develop a novel formulation for studying the large-amplitude free vibration behavior of uniform Timoshenko beams; slender beams forming a specific case when the slenderness ratio $\beta \rightarrow \infty$, with axially immovable ends, in which the nonlinearity exists in the strain-displacement relation of von Kármán type. In the present note, a single-term space mode approach adopted by many researchers is used, and the effect of the nonlinearity on the fundamental frequency is studied. It is interesting to note here that the proposed formulation is mainly based on the knowledge of two totally unrelated quantities like the axial tensile load developed in the beams due to large deflections [1] and the corresponding buckling load due to the end concentrated compressive loads. The authors emphasize that the novelty of the present formulation is that these two quantities mentioned have no direct relation to the study of linear and nonlinear free vibration behavior of structural members (beams), and is a major deviation from the earlier formulations discussed on this topic.

The present formulation is mainly based on the physical concepts of the problem and corresponding logical deductions with very much less mathematical treatment. The numerical results of the large-amplitude free vibrations of isotropic and uniform slender beams,

Table 1 Expressions for lateral displacement, tension, and buckling load parameters of Timoshenko beams

Serial no.	Beam boundary condition	Transverse displacement field w	Tension parameter λ_T	Buckling load parameter [12] λ_b
1	hinged–hinged beam	$b\sin\frac{\pi x}{L}$	$\frac{\pi^2}{4}(\frac{b_m}{r})^2$	$\frac{\pi^2}{1+(30.7932/\beta^2)}$
2	clamped–clamped beam	$b[1-\frac{\cos 2\pi x}{L}]$	$\frac{\pi^2}{4}(\frac{b_m}{r})^2$	$\frac{4\pi^2}{1+(123.1727/\beta^2)}$
3	hinged–clamped beam	$b[2x^4-3Lx^3+xL^3]$	$2.5382(\frac{b_m}{r})^2$	$\frac{7.2+(599.04/\beta^2)}{0.3429+(1869.01/\beta^4)+(44.9280/\beta^2)}$

with axially immovable ends and for various boundary conditions, are compared with those obtained using complex mathematical formulations [6–9] available in the literature, and are used to assess the accuracy, reliability, and consistency achieved using the present novel formulation. The variation of the nonlinear to linear radian frequencies for the first mode of vibration and the comparison of these with a more complex continuum formulation [10] is presented in this note for the Timoshenko beams.

Formulation

Following the work of the lead author [5], a formula of the form

$$\frac{\lambda_i}{\lambda_b}+\frac{\lambda_f}{\lambda_{f_o}}=1 \qquad (1)$$

can be derived for evaluating the frequency parameter λ_f of the initial axially loaded beams. The frequency and the load parameters are defined as $m\omega^2L^4/EI$ and PL^2/EI, respectively, with proper subscripts for ω and P. The assumption involved in deriving this formula is that the mode shapes of free vibration, with or without the initial axial load, and the buckling load parameters are exactly the same. This is satisfied when the single-term admissible functions are used for the lateral displacement w and its derivatives, depending on the boundary conditions. From this equation, if the values of λ_i, λ_b, and λ_{f_o} are known, one can easily evaluate λ_f without the repeated parametric solutions of initial axially loaded beams, where the parameters are λ_i/λ_b, b_m/r and β. The investigations presented [5] justify the use of Eq. (1), which is derived based on the assumption of simple harmonic motion (SHM) of the vibratory system.

Equation (1) can be effectively used to predict the large-amplitude free vibration behavior of beams as explained now. When the beams undergo large-amplitude free vibrations with axially immovable ends, an axial tensile load is developed for a given maximum amplitude b_m. The parameters λ_f and λ_{f_0} in Eq. (1) can be interpreted as λ_{NL_H}, where the subscript H denotes SHM, the nonlinear frequency parameter, λ_L ($b_m\to 0$), the linear frequency parameter and the initial load parameter λ_i as λ_T (tensile) arising due to the large deflections. With these interpretations, Eq. (1) can be written as

$$\frac{\lambda_{\mathrm{NL}_H}}{\lambda_L}=\frac{\omega^2_{\mathrm{NL}_H}}{\omega^2_L}=1+\frac{\lambda_T}{\lambda_b} \qquad (2)$$

Equation (2) can be used to obtain the expressions for the ratios of $\omega_{\mathrm{NL}_H}/\omega_L$ once the expressions for λ_T and λ_b are known. Equation (2) is valid if the vibratory motion is represented by SHM both in the linear and nonlinear regimes. But in the nonlinear regime, the assumption of SHM is not valid, and the results obtained for λ_{NL_H} (or ω_{NL_H}) are erroneous due to this assumption. As such, a correction for $\lambda_{\mathrm{NL}_H}/\lambda_L$ ($=\omega_{\mathrm{NL}_H}/\omega_L$) is necessary to obtain the accurate values of $\lambda_{\mathrm{NL}}/\lambda_L$ ($=\omega_{\mathrm{NL}}/\omega_L$) for a given maximum amplitude b_m. This correction is achieved through a correction factor and is obtained by applying the harmonic balance method (HBM) [11]. The correction factor evaluated from the HBM is three-quarters for cubic nonlinear homogeneous Duffing equation and modifies Eq. (2) as

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2=1+\frac{3}{4}\frac{\lambda_T}{\lambda_b} \qquad (3)$$

The expression for the axial tensile load developed because of the large deflections, taken from Woinowsky-Krieger [1], is

$$P_T=\frac{EI}{2Lr^2}\int_0^L\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\mathrm{d}x \qquad (4)$$

The admissible functions taken for w can be either trigonometric or algebraic, and have to necessarily satisfy the geometric boundary conditions. However, for better accuracy, the admissible functions can also be chosen to satisfy both the geometric and natural boundary conditions. The tensile load which is dependent on $(b_m/r)^2$ is obtained from Eq. (4). The buckling load parameter λ_b is evaluated, with the same functions chosen for w through the coupled displacement field formulation [12].

Numerical Results and Discussion

The three commonly used boundary configurations of the beams are considered in the present study and the numerical results are obtained for the maximum amplitude ratio b_m/r.

The single-term admissible functions with one undetermined coefficient for the lateral displacement w applicable for the first mode of vibration are presented in Table 1 in a concise form along with the other relevant information. To evaluate λ_T, the undetermined coefficient b has to be normalized with respect to the maximum w evaluated from the chosen admissible function, so that the value of the function at the maximum amplitude becomes unity. Throughout the present study, the value of υ and k are taken as 0.3 and 5/6, respectively.

For the Timoshenko beams, the expressions for the fundamental radian frequency ratio $\omega_{\mathrm{NL}}/\omega_L$ obtained using the present formulation based on the assumed single-term admissible functions, given in Table 1, are

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2=1+\frac{3}{16}\left[1+\frac{30.7932}{\beta^2}\right]\left(\frac{b_m}{r}\right)^2 \qquad (5)$$

for the hinged–hinged,

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2=1+\frac{3}{64}\left[1+\frac{123.1727}{\beta^2}\right]\left(\frac{b_m}{r}\right)^2 \qquad (6)$$

for the clamped–clamped, and

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2=1+\frac{3}{8}\frac{[1.7407+(228.0720/\beta^2)+(9487.8225/\beta^4)]}{[7.2+(599.04/\beta^2)]}\left(\frac{b_m}{r}\right)^2 \qquad (7)$$

for the hinged–clamped beams, where the maximum amplitude occurs at the point $0.4212L$ from the hinged end for the assumed displacement field.

These expressions are exactly matching with those obtained through the rigorous analysis based on the coupled displacement field concept [10], showing the simplicity and accuracy of the present novel formulation. The numerical results, in terms of $\omega_{\mathrm{NL}}/\omega_L$, for the three boundary configurations of the Timoshenko beams considered here for three slenderness ratios ($\beta=100$, 50, and 25), and for various maximum amplitude ratios b_m/r, are given in Table 2. From this table it can be seen that, for the Timoshenko beams ($\beta=50$ and 25) where the effects of shear deformation and rotary inertia are considerable, the values of $\omega_{\mathrm{NL}}/\omega_L$ match exactly with those obtained from the rigorous coupled displacement field concept [10]; the nonlinearity increasing with decreasing β. It may be noted here that $\beta=100$ represents almost a slender beam and the results match

Table 2 Variation of ω_{NL}/ω_L with b_m/r of Timoshenko beams for the first mode of vibration[a]

$\frac{b_m}{r}$	Hinged–hinged			Clamped–clamped			Hinged–clamped		
	25	50	100	25	50	100	25	50	100
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.0039	1.0038	1.0038	1.0011	1.0010	1.0009	1.0021	1.0019	1.0018
0.4	1.0156	1.0151	1.0149	1.0045	1.0039	1.0038	1.0085	1.0075	1.0073
0.6	1.0348	1.0336	1.0333	1.0100	1.0088	1.0085	1.0191	1.0169	1.0164
0.8	1.0611	1.0590	1.0585	1.0178	1.0156	1.0151	1.0338	1.0298	1.0289
1.0	1.0940	1.0905	1.0900	1.0277	1.0243	1.0235	1.0523	1.0462	1.0448
2.0	1.3368	1.3264	1.3237	1.1065	1.0940	1.0908	1.1954	1.1741	1.1689
3.0	1.6645	1.6457	1.6409	1.2268	1.2011	1.1946	1.4019	1.3607	1.3506
4.0	2.0366	2.0092	2.0023	1.3776	1.3368	1.3264	1.6481	1.5855	1.5701
5.0	2.4328	2.3969	2.3879	1.5501	1.4932	1.4786	1.9187	1.8345	1.8137

[a]The values presented here are exactly the same as given by Rao et al. [10]

very well with those of slender beams [6–9], though not given in this note for the sake of brevity.

Conclusions

The efficacy of the proposed novel formulation for studying the large-amplitude free vibration behavior of uniform Timoshenko beams, with hinged–hinged, clamped–clamped, and hinged–clamped boundary conditions, has been demonstrated in this note. The authors are of the opinion that the simplicity of the present formulation attracts both the analysts and researchers belonging to the basic field of nonlinear structural mechanics. In this note, the single-term admissible functions with one undetermined coefficient are used to represent the lateral displacement, corresponding to the boundary conditions considered, as most of the researchers on this topic concentrated on the first mode of vibration of slender beams using single-term admissible functions. The perfect match of the ratios of the nonlinear to linear radian frequencies obtained, through the rigorous coupled displacement field concept and the present novel formulations, shows that the present logically dominant formulation based on the physical concepts of the problem works beyond doubt for the Timoshenko beams. The formula derived for obtaining the frequency parameters of the initial axially loaded structural members is general, with an assumption on the mode shapes only. The present novel formulation which is developed for predicting the large-amplitude free vibrations, can be effectively applied to isotropic and composite beams and other structural members like circular (axisymmetric modes) and rectangular plates. However, proper values of the tension parameters developed due to large deflections along with the corresponding buckling load parameters are to be used in the formula developed.

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References

[1] Woinowsky-Krieger, S., "The Effect of an Axial Force on the Vibration of Hinged Bars," *Journal of Applied Mechanics*, Vol. 17, No. 1, 1950, pp. 35–36.

[2] Singh, G., Sharma, A. K., and Rao, G. V., "Large-Amplitude Free Vibrations of Beams: A Discussion Various Formulations and Assumptions," *Journal of Sound and Vibration*, Vol. 142, No. 1, 1990, pp. 77–85.

[3] Marur, R. S., "Advances in Nonlinear Vibration Analysis of Structures, Part 1: Beams," *Sadhana*, Vol. 26, Pt. 3, 2000, pp. 243–249.

[4] Satyamoorthy, M., *Nonlinear Analysis of Structures*, CRC Mechanical Engineering Series, CRC press, Boca Raton, FL, 1998.

[5] Rao, G. V., and Naidu, N. R., "Prediction of Fundamental Frequencies of Stressed Spring Hinged Tapered Beams," *AIAA Journal*, Vol. 39, No. 1, 2001, pp. 186–188.

[6] Rao, G. V., and Raju, K. K., "A Direct Numerical Integration Method to Study the Large Amplitude Vibration of Slender Beams with Immovable Ends," *Journal of the Institution of Engineers (India)*, Vol. 83, Nov. 2002, pp. 42–44.

[7] Singh, G., Rao, G. V., and Iyengar, N. G. R., "Re-Investigation of Large-Amplitude Free Vibrations of Beams Using Finite Elements," *Journal of Sound and Vibration*, Vol. 143, No. 2, 1990, pp. 351–355.

[8] Rao, G. V., and Raju, K. K., "Large Amplitude Free Vibrations of Beams: An Energy Approach," *ZAMM*, Vol. 83, No. 7, 2003, pp. 493–498.

[9] Raju, K. K., and Rao, G. V., "Towards Improved Evaluation of Large Amplitude Free Vibration Behavior of Uniform Beams Using Multiterm Admissible Functions," *Journal of Sound and Vibration*, Vol. 282, Nos. 3–5, 2005, pp. 1238–1246.

[10] Rao, G. V., Meera Saheb, K., and Ranga Janardhan, G., "Concept of Coupled Displacement Field for Large Amplitude Free Vibrations of Shear Flexible Beams," *Journal of Vibration and Acoustics*, Vol. 128, April 2006, pp. 251–255.

[11] Azrar, L., and White, R. G., "A Semi Analytical Approach to the Nonlinear Dynamic Response Problems of S-S and C-C Beams at Large Vibration Amplitudes, Part 1: General Theory and Application to the Single Mode Approach to Free and Forced Vibration Analysis," *Journal of Sound and Vibration*, Vol. 224, No. 2, 1999, pp. 183–207.

[12] Meera Saheb, K., Rao, G. V., and Ranga Janardhan, G., "Coupled Displacement Field Formulation for the Buckling Analysis of Shear Flexible Columns," *Journal of Structural Engineering (India)*, Vol. 33, No. 5, Dec. 2006–Jan. 2007, pp. 413–418.

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